

Implementation of the Graded Ring of Quasimodular Forms

Sagemath Project

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The software SageMath is known for its implementation of (classical) modular forms. However, the software does not support the notion of *quasimodular forms* (also called *quasiforms*), which are an extension of the classical notion of modular forms. An example of a quasimodular form is the *weight 2 Eisenstein series* E_2 . In fact, it is possible to show that any quasimodular form can be obtained using operations (addition, multiplications) between the classical modular forms and the quasiform E_2 . The study of quasimodular forms is interesting because they are related to combinatorial and algebraic objects through the Bloch and Okounkov theorem. This theorem asserts that one can produce a quasimodular form using a certain type of polynomial (called a *shifted symmetric polynomial*). The computational nature of this theorem suggest that it would be possible to implement it in SageMath. Hence, the goal of this project is twofold: to implement the space of quasimodular forms and to implement the Block-Okounkov bracket that, given a shifted symmetric polynomial, produces a quasimodular form.

To gain some perspective on the project, I created the trac ticket [#31512](#) and implemented an initial version of the space of quasimodular forms. This initial implementation still lacks a lot of features but strongly suggests that the project is realizable. After discussion with the mentor Vincent Delecroix, a suggestion was made to separate the goal of the project in multiple tasks:

1. **[#31559](#): Make the class `ModularFormsRing` manipulates formal object.**

In the study of modular forms, one observes that to each modular form we can associate a positive number, called the *weight* of the modular forms. At the moment, SageMath only supports operations between modular forms of equal weight. Thus, the goal of this task is to implement the classical operations (addition, multiplication) between mixed-weight modular forms.

2. **[#31512](#): Implement the graded ring of quasimodular forms.**

The goal here is to implement the space of quasimodular forms. The idea behind the implementation is to define a quasimodular form as a polynomial in E_2 (the weight 2 Eisenstein series) with coefficients in the space of classical modular forms. With this implementation, it should be possible to perform the classical operations between two elements of the space. Hence, this task depends of the first task.

3. Implement the Bloch-Okounkov bracket.

The Bloch-Okounkov bracket associate a *power series in $\mathbb{Q}[[q]]$* (i.e., an infinite sequence) to any function $f: P \rightarrow \mathbb{Q}$, where P is the set of all partitions. A theorem by Bloch-Okounkov states that the bracket evaluated at a special class of functions, called the *shifted symmetric polynomials*, correspond to a quasimodular form. Thus, the goal of this task is to implement this bracket. In order to achieve this, we will first need to implement the shifted symmetric polynomials into SageMath.

These three tasks are also summarized in the trac ticket [#31560](#) (after discussion with V. Delecroix). Even though I already started to work on the second task in the past weeks, it is intended to clear these tasks in the order written above, since some components depend on one another.

Now, there are two possible problems that could occur during this project. First, there could be some functionalities that are not implemented yet in SageMath that we are not aware of. To work around this possible problem, it is important to study the source code in depth and understand the tools that are at our disposal. If this issue occurs, we will create a new trac ticket and implement the missing functionality. Second, during the implementation of the first task (which is the support of operations between mixed weight modular forms), there could be some components in SageMath that are dependent of the old implementation. Hence, in order to not introduce any unwanted bug, it will be essential to test the software completely.

In conclusion, the main goal of this project is to implement two new features in SageMath, namely the space of quasimodular forms and the Bloch-Okounkov bracket. To reach this goal, it will be important to follow the three tasks above. The computational nature of quasimodular forms, the initial implementation in ticket [#31512](#) and the anticipated solutions for the most probable issues give us great hope that the project will be realizable in the ten weeks timeframe.

Internet Links

- Github page: github.com/DavidAyotte
- Sage Trac Server: trac.sagemath.org
- Trac tickets:
 - [#31559](https://trac.sagemath.org/ticket/31559): trac.sagemath.org/ticket/31559
 - [#31512](https://trac.sagemath.org/ticket/31512): trac.sagemath.org/ticket/31512
 - [#31560](https://trac.sagemath.org/ticket/31560): trac.sagemath.org/ticket/31560